Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00126-X

J. Appl. Maths Mechs, Vol. 64, No. 6, pp. 951–957, 2000

0021–8928/00/$—see front matter

THE EFFECTIVE CONDUCTIVITY OF A RANDOMLY INHOMOGENEOUS MEDIUM†

E. V. TEODOROVICH

Moscow

e-mail: teodor@ipmnet.ru

(*Received* 19 *November* 1999)

Using a representation of the solution to the diffusion equation in a randomly inhomogeneous medium in the form of a Feynman path integral an explicit expression is obtained for the effective conductivity in a space of arbitrary dimension. A calculation of the path integral only turns out to be possible in the case of a large-scale limit. In particular, it is shown that in the three-dimensional case the expression for the effective conductivity does not admit of an expansion in terms of the conductivity variance. This indicates that the use of standard perturbation theory in the form of an expansion in terms of the conductivity fluctuations is incorrect.

1. INTRODUCTION

When analysing various phenomena of the transport of some quantity (charge, heart, material or momentum) it is usually assumed that the relation between the flows of the corresponding quantities and the gradients of some other quantities (potential, temperature, concentration or velocity) is linear, while in a randomly inhomogeneous medium the constant of proportionality (referred to as the transport coefficient) is a random function of the coordinates. In practice, it is important to know the relation between the averaged flows and the gradients. The relevant constant of proportionality is termed the effective permeability. Though our analysis concerns many different phenomena, to be more specific we shall examine the process of seepage flow in porous media.

The description of seepage processes in an isotropic medium is based on Darcy's law, which relates the seepage flow rate $\mathbf{v}(\mathbf{r})$ and the pressure gradient $p(\mathbf{r})$

$$\mathbf{v}(\mathbf{r}) = -\varkappa(\mathbf{r})\nabla p(\mathbf{r}) \tag{1.1}$$

where $\varkappa(\mathbf{r})$ is the conductivity of the medium, which is a given function of the coordinate $\mathbf{r}$. In general, the relation between the mean values of the seepage flow rate and the pressure gradient is non-local and has the form

$$\langle \mathbf{v}(\mathbf{r})\rangle = -\int \varkappa_{\text{eff}}(\mathbf{r},\mathbf{r}')\langle \nabla p(\mathbf{r}')\rangle d\mathbf{r}' \tag{1.2}$$

For a statistically homogeneous medium the integral kernel $\varkappa_{\text{eff}}(\mathbf{r},\mathbf{r}')$ depends only on the difference of the coordinates $\mathbf{r}-\mathbf{r}'$. When the inhomogeneity dimensions are small compared with the characteristic scales of the seepage flow (the large-scale limit), we may assume

$$\varkappa_{\text{eff}}(\mathbf{r}-\mathbf{r}') = \varkappa_{\text{eff}}\delta(\mathbf{r}-\mathbf{r}'), \quad \varkappa_{\text{eff}} \equiv \int \varkappa_{\text{eff}}(\mathbf{r})d\mathbf{r} \tag{1.3}$$

and the problem consists of finding the effective conductivity $\varkappa_{\text{eff}}$ for given statistics of the forms of conductivity $\varkappa(\mathbf{r})$. It is also worth noting that, according to the convolution theorem, in the space of Fourier-transforms relation (1.2) takes the form

$$\langle \mathbf{v}(\mathbf{q})\rangle = -i\mathbf{q}\varkappa_{\text{eff}}(\mathbf{q}) < p(\mathbf{q})\rangle$$

and relation (1.3) will correspond to

$$\varkappa_{\text{eff}} = \varkappa_{\text{eff}}(\mathbf{q})|_{q=0} \tag{1.4}$$

A calculation of the effective conductivity in a randomly inhomogeneous medium has been the subject

†*Prikl. Mat. Mekh.* Vol. 64, No. 6, pp. 989–995, 2000.

of many papers. The main difficulty is that, in order to find the average pressure and flow rates, one has to start by solving the following stochastic differential equation, which follows from the condition that the liquid is incompressible

$$\nabla \cdot (\varkappa(\mathbf{r})\nabla p(\mathbf{r})) = 0$$

for given boundary conditions (usually deterministic), and the next step is to average the solution obtained over the ensemble of different conductivities $\varkappa(\mathbf{r})$. Since, in general case, the solution of a differential equation with variable coefficients cannot be written in explicit form, it seems best to use an approach based on perturbation theory. To do this one must split $\varkappa(\mathbf{r})$ into an averaged part, which is independent of coordinates in the statistically homogeneous medium, and fluctuations

$$\varkappa(\mathbf{r}) = \langle\varkappa\rangle + \delta\varkappa(\mathbf{r})$$

and search for a solution in the form of a perturbation theory series in powers of the fluctuations $\delta\varkappa(\mathbf{r})/\langle\varkappa\rangle$. Then term-by-term averaging of the series obtained for known statistics of the fluctuations $\delta\varkappa$ enables one to find the mean values of the flow rates and pressures and thus calculate $\varkappa_{\text{eff}}$. A detailed description of the approach based on the use of a low-order perturbation theory approximation is given in the monograph [1]; most of the papers devoted to this problem use this approach.

However, the question of to what extent the effective conductivity is well described by the low-order approximation remains unanswered. In the context of estimating the role of high-order approximations the suggestion was made that the dependence of the effective conductivity on the conductivity variance has an exponential form and that the lowest-order approximation is the first term of the Taylor series expansion of the exponential function [2]. To check this hypothesis, the high-order perturbational corrections were estimated. In particular, the result obtained in the second-order approximation (quadratic in the conductivity variance) turned out to be in agreement with the hypothesis that the effective conductivity depends exponentially on the variance [3]. Unlike the "primitive" perturbation theory used in [1, 4] it was suggested that the perturbation theory series should be constructed by proceeding from the differential equation to the integral one and then solving that by an iterative method [5–7]. In this approach it turned out to be very useful to employ quantum field theory methods based on the use of Feynman diagrams, Dysons' equations and the renormalization technique for improving perturbation theory by summing some infinite subsequence of the total perturbation series [5]. The results for high-order approximations obtained within the scope of the improved perturbation theory turned out to contradict the hypothesis of an exponential dependence of the conductivity on the variance as well as the assumption that the effective conductivity is independent of the form of the correlation function of the conductivity fluctuations [4].

It is of interest to calculate the effective conductivity without using perturbation theory. A similar approach was suggested by us in the problem of the turbulent diffusion of a passive impurity [8]. The approach is based on representing the solution of the stochastic differential equation in the form of a Feynman path integral and does not use the assumption that the conductivity fluctuations are small [10].

2. STATEMENT OF THE PROBLEM

In order to set up the boundary-value problem we will consider an unbounded medium with a given regular source rather than the generally used method when the boundary pressure is given [1]. Then the equations for the flow rate and pressure take the form

$$\nabla \mathbf{v}(\mathbf{r}) = \rho(\mathbf{r}) \tag{2.1}$$

$$\nabla \varkappa(\mathbf{r}) \cdot \nabla p(\mathbf{r}) = -\rho(\mathbf{r}) \tag{2.2}$$

Here $\rho(\mathbf{r})$ is the liquid source density. We will assume (and this will be confirmed below) that the result for the conductivity, which specifies the properties of the medium, must be independent of the geometry of the flows produced by them.

The solution of Eq. (2.1) for the potential part of the flow is

$$\langle\mathbf{v}(\mathbf{r})\rangle = \nabla\Delta^{-1}\rho(\mathbf{r})$$

where Δ^{-1} is Green's function for the Laplace operator. Taking Fourier transforms and averaging we obtain

$$\langle \mathbf{v}(\mathbf{q})\rangle = -i\mathbf{q}\rho(\mathbf{q})/q^2$$

As a result, for the Fourier transform of the effective conductivity we find

$$\varkappa_{\text{eff}}^{-1}(\mathbf{q}) = q^2\langle p(\mathbf{q})\rangle / \rho(\mathbf{q}) \tag{2.3}$$

and the problem reduces to calculating the Fourier transform of the pressure averaged over the ensemble of conductivities.

In what follows, it turns out to be more convenient to introduce the notation $\varkappa(\mathbf{r}) = \varkappa_0 e^{u(\mathbf{r})}$. The equation for pressure then takes the form

$$[\nabla^2 + \nabla u(\mathbf{r})\cdot\nabla]p(\mathbf{r}) = -\varkappa_0^{-1}e^{-u(\mathbf{r})}\rho(\mathbf{r}) \tag{2.4}$$

It should be noted that when constructing the perturbation series for the pressure, the exponential factor on the right-hand side of Eq. (2.4) was ignored in papers based on the use of an external source [5–7].

The formal solution of Eq. (2.4) may be determined by the relation

$$p = \varkappa_0^{-1}(-\nabla^2 - \nabla u\cdot\nabla)^{-1}e^{-u}\rho \tag{2.5}$$

where the inverse operator in (2.5) is Green's function for an equation with variable random coefficients.

3. THE STOCHASTIC SOLUTION FOR THE PRESSURE

Following the approach described previously [8, 9] the Feynman operator formalism [10] will be used to construct the inverse operator. According to that formalism the inverse operator may be expressed in the form of the ordered operator exponential function

$$[-\nabla^2 - \nabla u(\mathbf{r})\cdot\nabla]^{-1} = \int_0^\infty \exp\left\{-\int_0^\tau [-\nabla^2(s) - \nabla u(\mathbf{r},s)\cdot\nabla(s)]ds\right\}d\tau \tag{3.1}$$

Here the exponential function of operator should be understood as a Taylor series expansion in powers of the operators $\nabla(s)$ and $\nabla\, u(\mathbf{r}, s)$ provided the requirement that non-commuting operators $\nabla(s)$ and $\nabla\, u(\mathbf{r}, s)$ act in order of increasing "proper time" s. This ordering rule enables one to manipulate operators as numbers. To "disentangle the operator exponential function the order" of the operator ∇ in the exponent of the exponential should be reduced to the first power. Then the exponential function obtained, which contains the operator ∇ to the first power, can be interpreted as the shift operator of the argument according to the relation

$$e^{\mathbf{b}\cdot\nabla}f(\mathbf{r}) = f(\mathbf{r}+\mathbf{b}) \tag{3.2}$$

The reduction of the order of the operator ∇ may be achieved by using a transformation proposed by Stratonovich [11]. This transformation must be regarded as the functional analogue of the Weierstrass transformation [12] (see also [9, Appendix A])

$$\exp\left\{a\int_0^\tau \mathbf{F}^2(s)ds\right\} = \int\exp\left\{-\int_0^\tau\left[\frac{1}{4a}\mathbf{X}^2(s) + \mathbf{F}(s)\cdot\mathbf{X}(s)\right]ds\right\}d[\mathbf{X}(s)] \tag{3.3}$$

Here $\mathbf{X}(s)$ are all possible vector functions, specified on the interval $(0, \tau)$. The integral measure $d[\mathbf{X}(s)]$ is normalized in such a way that

$$\int\exp\left\{-\frac{1}{4a}\int_0^\tau \mathbf{X}^2(s)ds\right\}d[\mathbf{X}(s)] = 1$$

Performing Stratonovich's transformation and using relation (3.2) we obtain

$$p(\mathbf{r}) = \varkappa_0^{-1} \int_0^{\infty} \int \rho(\mathbf{r}(0,\tau\,|\,\mathbf{X}))\times$$
$$\times\exp\left\{-\frac{1}{4}\int_0^{\tau} [\mathbf{X}(s)+\nabla u(\mathbf{r}(s,\tau\,|\,\mathbf{X}))]^2 ds - u(\mathbf{r}(0,\tau\,|\,\mathbf{X}))\right\} d[\mathbf{X}(s)]d\tau \tag{3.4}$$
$$\mathbf{r}(s,\tau\,|\,\mathbf{X}) = \mathbf{r} - \int_s^{\tau} \mathbf{X}(s')ds'$$

To perform subsequent averaging over samples of $u(\mathbf{r}) = \ln[\varkappa(\mathbf{r})/\varkappa_0]$, it is convenient to eliminate the squared ∇u in the exponent of the exponential function by a second application of Stratonovich's transformation. As a result we obtain the following for the average value of the pressure

$$\langle p(\mathbf{r})\rangle = \varkappa_0^{-1} \int_0^{\infty} \int \rho(\mathbf{r}(0,\tau\,|\,\mathbf{X}))\times$$
$$\times\exp\left\{-\int_0^{\tau} [\mathbf{Y}^2(s)+i\mathbf{Y}(s)\cdot\mathbf{x}(s)]ds\right\}\Psi[\theta(\mathbf{x}\,|\,\mathbf{Y})]d[\mathbf{X}(s)]d[\mathbf{Y}(s)]d\tau \tag{3.5}$$
$$\Psi[\theta(\mathbf{x}\,|\,\mathbf{Y})] = \left\langle \exp\left\{i\int \theta(\mathbf{x}\,|\,\mathbf{Y})u(\mathbf{x})d\mathbf{x}\right\}\right\rangle$$
$$\theta(\mathbf{x}\,|\,\mathbf{Y}) = \int_0^{\tau} [\mathbf{Y}(s)\cdot\nabla + 2i\delta(s)]\delta(\mathbf{x}-\mathbf{r}(s,\tau\,|\,\mathbf{X}))ds$$

The quantity $\Psi[\theta(\mathbf{x}\,|\,\mathbf{Y})]$ is the characteristic functional for the conductivity logarithm (the functional Fourier-transform of the distribution function for $u(\mathbf{r})$), which in the case of a centred log-normal conductivity distribution has the form

$$\Psi[\theta(\mathbf{x}\,|\,\mathbf{Y})] = \exp\left\{-\frac{1}{2}\int \theta(\mathbf{x}\,|\,\mathbf{Y})\theta(\mathbf{x}'\,|\,\mathbf{Y})B(\mathbf{x},\mathbf{x}')d\mathbf{x}d\mathbf{x}'\right\} \tag{3.6}$$

where $B(\mathbf{x} - \mathbf{x}') = \langle u(\mathbf{x})u(\mathbf{x}')\rangle$ is the binary correlation function of the conductivity logarithm. By a suitable choice of $\varkappa_0$ one may always satisfy the condition $\langle u\rangle = 0$.

After integrating over $\mathbf{x}$, $\mathbf{x}'$ in (3.6) the dependence on $\mathbf{r}$ remains only as an independent variable of the source density. Performing a Fourier transformation with respect to $\mathbf{r}$ and using (2.3) we find

$$\varkappa_{\text{eff}}^{-1}(\mathbf{q}) = \varkappa_0^{-1} q^2 \int_0^{\infty} \int \exp\left\{-\int_0^{\tau} [\mathbf{Y}^2(s)+ + i\mathbf{Y}(s)\cdot\mathbf{X}(s) + i\mathbf{q}\cdot\mathbf{X}(s)]ds\right\}\times$$
$$\times E(\tau|\mathbf{Y})d[\mathbf{X}(s)]d[\mathbf{Y}(s)]d\tau \tag{3.7}$$
$$E(\tau\,|\,\mathbf{Y}) = \exp\left\{-\frac{1}{2}\int_0^{\tau}\int_0^{\tau} [-\mathbf{Y}(s)\nabla + 2i\delta(s)][\mathbf{Y}(s')\nabla + 2i\delta(s')]\times\right.$$
$$\left.\times B(\mathbf{r}(0,s\,|\,\mathbf{X}) - \mathbf{r}(0,s'\,|\,\mathbf{X}))dsds'\right\}$$

To find $\varkappa_{\text{eff}}$, we take the limit as $q \to 0$. However, this operation is not simple because the dependence of the integrals on the right-hand side of relation (3.7) on q proves to be singular at the point $q = 0$. To make this operation well posed, we perform an inverse Fourier transformation and we then obtain

$$\varkappa_{\text{eff}}^{-1}(\mathbf{r}) = -\varkappa_0^{-1}\nabla^2 F(r) \tag{3.8}$$
$$F(r) = \int_0^{\infty} \int \delta(\mathbf{r}(0,\tau\,|\,\mathbf{X}))\exp\left\{-\int_0^{\tau} [\mathbf{Y}^2(s) + i\mathbf{Y}(s)\cdot\mathbf{X}(s)]ds\right\}\times$$
$$\times E(\tau|\mathbf{Y})d[\mathbf{X}(s)]d[\mathbf{Y}(s)]d\tau$$

According to (3.4) and (3.8) the integration over $\mathbf{X}(s)$ may be regarded as an integration over the velocities of all possible paths emerging from the origin of coordinates at the time $s = 0$ and ending at the point $\mathbf{r}$ at the time $s = \tau$.

4. THE EFFECTIVE CONDUCTIVITY IN THE LARGE-SCALE LIMIT

Using relations (1.3) and (3.8) and transforming the volume integral into a surface integral we obtain

$$\varkappa_{\text{eff}}^{-1} = -\varkappa_0^{-1} S_d \lim_{r\to\infty} r^{d-1} \frac{\partial}{\partial r} F(r) \tag{4.1}$$

($S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the d-dimensional sphere of unit radius).

When integrating over trajectories $\mathbf{X}(s)$, the contribution of any trajectory depends exponentially on the quantity $\int_0^\tau \mathbf{X}^2(s)ds$, which can be interpreted as a twice the action along the trajectory of a moving free particle. According to relation (4.1) only trajectories departing to infinity are considered, hence the action along these trajectories will be large, and the most essential contribution to the path integral results from trajectories with minimum action, e.g. with constant velocity $\mathbf{X}(s) = \mathbf{X}_0 = \text{const}$.

$$\int_0^\tau \mathbf{X}^2(s)ds$$

With the exception of the assumption of the log-normal form of the conductivity distribution, our analysis has so far been exact. As the first (and unique) approximation we assume that in Eq. (3.8) it is possible to put $B(\mathbf{r}(s,\tau|\mathbf{X}) - \mathbf{r}(s',\tau|\mathbf{X})) \approx B(\mathbf{X}_0(s-s'))$.

In this approximation and using the identity

$$\nabla B(\mathbf{X}_0(s-s')) = \frac{\mathbf{X}_0}{\mathbf{X}_0^2}\frac{\partial}{\partial s} B(\mathbf{X}_0(s-s')) = -\frac{\mathbf{X}_0}{\mathbf{X}_0^2}\frac{\partial}{\partial s'} B(\mathbf{X}_0(s-s'))$$

one can integrate over s and s' in Eq: (3.8). Integration over trajectories is also possible in explicit form using a Fourier transform [13]. In order to do this, we perform a Fourier series expansion of the functions $\mathbf{X}(s)$ and $\mathbf{Y}(s)$, given in the interval $0 \leqslant s \leqslant \tau$;

$$\mathbf{X}(s) = \mathbf{X}_0 + \sum_{n=1}^{\infty} \mathbf{X}_n \cos\frac{\pi n s}{\tau}, \quad \mathbf{Y}(s) = \mathbf{Y}_0 + \sum_{n=1}^{\infty} \mathbf{Y}_n \cos\frac{\pi n s}{\tau} \tag{4.2}$$

Using these expansions (which in the case of the path integral corresponds to a simple change of the variable of integration), the integration over trajectories reduces to the product of an infinite number of integrals over $\mathbf{X}_n$ and $\mathbf{Y}_n$. Integrating over $\mathbf{X}_n$, if $n \neq 0$, gives $\delta(\mathbf{Y}_n)$, and taking into account the normalization condition the subsequent integration over $\mathbf{Y}_n$ gives a factor equal to unity. After substituting $\mathbf{Y}_0 \to \mathbf{Y}_0 - i\mathbf{X}_0/2$ the remaining integration over d-dimensional vector $\mathbf{Y}_0$ is performed using the relation

$$\int \exp\left\{-\mathbf{Y}^2\tau - \frac{(\mathbf{Y}\cdot\mathbf{X})^2}{\mathbf{X}^4}[B(0) - B(\mathbf{X}\tau)]\right\} d\mathbf{Y} = \left(\frac{\pi}{\tau}\right)^{d/2} \left[1 + \frac{B(0) - B(\mathbf{X}\tau)}{\mathbf{X}^2\tau}\right]^{-1/2}$$

The subsequent integration over $\mathbf{X}_0$ is carried out using the delta-function in the integrand of (3.8).

5. DISCUSSION OF THE RESULTS

After performing the actions indicated in Section 4 and using the relation $\mathbf{X}_0 = r/\tau$ we obtain

$$\varkappa_{\text{eff}}^{-1} = \varkappa_0^{-1} S_d \frac{\exp\{B(0)/2\}}{(4\pi)^{d/2}} \lim_{r\to\infty} r^{d-1} \frac{\partial}{\partial r} \int_0^\infty \frac{\exp\{-r^2/4\tau\}}{\sqrt{1 + [B(0) - B(r)]r^2/\tau}} \frac{d\tau}{\tau^{d/2}} =$$

$$= \varkappa_0^{-1} S_d \frac{\exp\{B(0)/2\}}{(4\pi)^{d/2}} \lim_{r\to\infty} \left[(d-2)A_1(r) + \frac{1}{2} r \frac{\partial B(r)}{\partial r} A_3(r)\right] \tag{5.1}$$

$$A_n(r) = \int_0^\infty \frac{t^{(d-3)/2} \exp\{-t/4\}}{[t + B(0) - B(r)]^{n/2}} dt$$

According to the principle of damping of the correlations we assume that the correlation $B(r) \to 0$ and $\partial B(r)/\partial r$ tends to zero more rapidly then $1/r$ as $r \to \infty$. As the result we find

$$\varkappa_{\text{eff}}^{-1} = \varkappa_0^{-1} \frac{(d-2)S_d \exp\{B(0)/2\}}{(4\pi)^{d/2}} \int_0^\infty \frac{t^{(d-3)/2} \exp\{-t/4\}}{\sqrt{t + B(0)}} dt \qquad (5.2)$$

Notice that if we put $B(0) = 0$ in the integrand of (5.2), the integral in (5.2) will be equal to $(4)^{d/2-1}\Gamma(d/2 - 1)$, and this leads to the relation $\varkappa_{\text{eff}}^{-1} = \varkappa_0^{-1} \exp\{B(0)/2\} = \langle \varkappa^{-1} \rangle$. In the one-dimensional case this result is exact, whereas a direct application of relation (5.2) leads to a singular integral because in the one-dimensional case the use of the Fourier transform proves to be incorrect, as is well-known when solving the Laplace equation by means of the integral transform method. For $d = 2$, formal use of relation (3.2) gives $\varkappa_{\text{eff}} = 0$. This result is related to the classical problem of finding Green's function for the d-dimensional Laplace equation when the solution has the form $\Gamma(d/2 - 1)r^{-(d-2)}$, and as $d \to 2$, it turns out to be expressed in the form $\Gamma(0) + 2 \ln r$, which contains an infinite constant and a finite term proportional to $\ln r$

One can also see from formulae (5.2) that the right-hand side of (5.2) does not allow of a series expansion in terms of the variance $B(0)$ because, beginning from some order, the expansion coefficients contain divergent integrals. This enables us to assert that when constructing the statistical solution of a stochastic differential equation of the form (2.2), the use of the perturbation technique is incorrect. In other words, the perturbation proves to be singular, and even under very small perturbations the structure of the solution is not close to the structure of the solution of the unperturbed problem. It seems likely that the singular nature of the perturbation is associated with a violation of the symmetry group corresponding to the scale transformation of the coordinates. This symmetry exists for Eq. (2.2), if $\varkappa = const$, and does not exist if the conductivity depends on the coordinate.

The above is clearly illustrated in the three-dimensional case where the problem of divergent integrals does not arise and the result of the calculation may be presented in explicit form. In the general case, the integral in (5.2) is expressed in terms of the Whittaker function $W_{-d/4+1/2, d/4-1/2}(B(0)/4)$ which, when $d = 3$, reduces to the error integral. The appropriate calculation yields

$$\varkappa_{\text{eff}} = \frac{\langle \varkappa \rangle}{(1 + D)[1 - \operatorname{erf}(\sqrt{B(0)}/2)]}$$

$$D + 1 = \frac{\langle \varkappa^2 \rangle}{\langle \varkappa \rangle^2} = e^{B(0)}$$

and the series expansion in terms of the variance of the conductivity fluctuations is carried out in half-integer powers of $B(0)$, whereas from the procedure for constructing the perturbation series it follows that we have to obtain an expansion in integer powers of $B(0)$. However, the question of to what extent the conclusion on the non-applicability of perturbation theory is connected with the use of the log-normal statistics for the conductivity fluctuations remains open.

This research was partially financed by the Russian Foundation for Basic Research (99-01-01083 and 99-01-00957) and was reported at the International Conference "Modern Approaches to Flows in Porous Media" (September 1999) dedicated to the memory of P. Ya. Polubarinova-Kochina.

REFERENCES

1. SCHWIDLER, M. I., *Stochastic Hydrodynamics of Porous Media*. Nedra, Moscow, 1985.
2. GELHAR, L. W., and AXNESS, C. L., Three-dimensional stochastic analysis of macrodispersion in aquifers. *Water Resour. Res.*, 1983, **19**, 1, 161–180.
3. DAGAN, G., Higher-order correction of effective permeability of heterogeneous formations of log-normal conductivity distribution. *Trans. Porous Media*, 1993, **12**, 3, 279–290.
4. De WIT, A., Correlation structure dependence of the effective permeability of heterogeneous porous media. *Phys. Fluids*, 1995, **7**, 11, 2553–2562.
5. KING, P. R., The use of field theoretic methods for the study of flow in a heterogeneous porous media. *J. Phys. A: Math. Gen.*, 1987, **20**, 12, 3935–3947.

6. INDELMAN, P. and ABRAMOVICH, B., A higher-order approximation to effective conductivity in media of anisotropic random structure. *Water Resour. Res.*, 1994, **30**, 6, 1857–1864.
7. ABRAMOVICH, B. and INDELMAN, P., Effective permeability of log-normal random media. *J. Phys. A.*, 1995, **28**, 3, 693–700.
8. TEODOROVICH, E. V., Turbulent diffusion of a passive impurity. *Prikl. Mat. Mekh.*, 1992, **55**, 2, 275–282.
9. TEODOROVICH, E. V., Calculation of the effective permeability of randomly inhomogeneous porous medium. *Zh. Epsp. Teor. Fig.*, 1997, **112**, 1, 313–324.
10. FEYNMAN, R. P., An operator calculus having applications to quantum electrodynamics. *Phys. Rev.*, 1951, **84**, 1, 118–128.
11. STRATONOVICH, R. L., A method of calculating quantum distribution functions. *Dokl. Akad. Nauk. SSSR*, 1957, **115**, 6. 1097–1100.
12. HIRSCHMAN, I. I. and WIDDER, D. V., *The Convolution Transform*. Princeton University Press, NJ, 1955.
13. FEYNMAN, R. P. and HIGGS, A. R., *Quantum Mechanics and Path Integrals*. McGraw-Hill, New York, 1956.

Translated by the author